

# JACOB'S LADDERS, HETEROGENEOUS QUADRATURE FORMULAE, BIG ASYMMETRY AND RELATED FORMULAE FOR THE RIEMANN ZETA-FUNCTION

JAN MOSER

ABSTRACT. In this paper we obtain as our main result new class of formulae expressing correlation integrals of the third-order in  $Z$  on disconnected sets  $\check{G}_1(x), \check{G}_2(y)$  by means of an autocorrelative sum of the second order in  $Z$ . Moreover, the distance of the sets  $\check{G}_1(x), \check{G}_2(y)$  from the set of arguments of autocorrelative sum is extremely big, namely  $\sim A\pi(T)$ ,  $T \rightarrow \infty$ , where  $\pi(T)$  is the prime-counting function.

## 1. INTRODUCTION

1.1. Let us remind the following formulae

$$(1.1) \quad \begin{aligned} \int_T^{2T} Z^4(t) dt &\sim \frac{2\pi}{\ln T} \sum_{T \leq t_\nu \leq T+2T} Z^4(t_\nu), \quad T \rightarrow \infty, \\ \int_T^{T+U} Z^2(t) dt &\sim \frac{2\pi}{\ln T} \sum_{T \leq t_\nu \leq T+U} Z^2(t_\nu), \quad U = \sqrt{T} \ln T, \quad T \rightarrow \infty \end{aligned}$$

(see [5], (4.4), (4.7)), where

$$(1.2) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right) \end{aligned}$$

(see [9], pp. 79, 329), and  $\{t_\nu\}$  is the Gram sequence (comp. [9], p. 99). The formulae (1.1) were proved by us in connection with the Kotelnikov-Whittaker-Nyquist theorem. Namely, by these formulae we have expressed the biquadratic and quadratic effects for the continuous signals

$$Z(t), \quad t \in [T, 2T]; \quad t \in [T, T+U]$$

from the point of view of information theory.

1.2. The formulae (1.1) are:

- (a) asymptotic quadrature formulae (from the left to the right),
- (b) asymptotic summation formulae (from the right to the left).

Next, for the second formula in (1.1) (for example) we have

$$(1.3) \quad \begin{aligned} t \in [T, T+U] &\rightarrow t_\nu \in [T, T+U], \\ Z^2(t) &\rightarrow Z^2(t_\nu), \end{aligned}$$

---

*Key words and phrases.* Riemann zeta-function.

i. e. the segment  $[T, T + U]$  and the exponent 2 are conserved.

*Remark 1.* By (1.3) it is natural to call the formulae of the kind (1.1) *homogeneous formulae*.

On the contrary, we obtain in this paper some heterogeneous asymptotic quadrature formulae – such formulae that the properties (1.3) are not fulfilled.

## 2. HETEREGENEOUS QUADRATURE FORMULAE

2.1. Let (see [3], (3))

$$\begin{aligned}
 G_1(x) &= G_1(x; T, H) = \\
 &= \bigcup_{T \leq t_{2\nu} \leq T+H} \{t : t_{2\nu}(-x) < t < t_{2\nu}(x)\}, \quad 0 < x \leq \frac{\pi}{2}, \\
 (2.1) \quad G_2(y) &= G_2(y; T, H) = \\
 &= \bigcup_{T \leq t_{2\nu+1} \leq T+H} \{t : t_{2\nu+1}(-y) < t < t_{2\nu+1}(y)\}, \quad 0 < y \leq \frac{\pi}{2}, \\
 H &= T^{1/6+2\epsilon},
 \end{aligned}$$

where the collection of sequences

$$\{t_\nu(\tau)\}, \quad \tau \in [-\pi, \pi], \quad \nu = 1, 2, \dots$$

is defined by the equation (see [3], (1))

$$\vartheta[t_\nu(\tau)] = \pi\nu + \tau; \quad t_\nu(0) = t_\nu,$$

and (see [3], (8))

$$(2.2) \quad m\{G_1(x)\} = \frac{x}{\pi}H + \mathcal{O}\left(\frac{x}{\ln T}\right), \quad m\{G_2(y)\} = \frac{y}{\pi}H + \mathcal{O}\left(\frac{y}{\ln T}\right),$$

where  $m\{G_1(x)\}, \dots$  is the measure of the set  $G_1, \dots$ .

Let

$$\varphi_1\{\mathring{G}_1(x)\} = G_1(x), \quad \varphi_1\{\mathring{G}_2(y)\} = G_2(y).$$

The following Theorem holds true.

**Theorem 1.**

$$\begin{aligned}
 (2.3) \quad & \int_{\mathring{G}_1(x)} \omega(t) Z[\varphi_1(t)] Z^2(t) dt = \\
 &= \frac{m\{G_1(x)\}}{Q_1 \ln P_0} \sum_{T \leq t_\nu \leq T+U_1} Z(t_\nu) Z\left(t_\nu + \frac{x}{\ln P_0}\right) + \mathcal{O}\left(\frac{H}{\ln T}\right), \\
 & \int_{\mathring{G}_2(y)} \omega(t) Z[\varphi_1(t)] Z^2(t) dt = \\
 &= -\frac{m\{G_2(y)\}}{Q_1 \ln P_0} \sum_{T \leq t_\nu \leq T+U_1} Z(t_\nu) Z\left(t_\nu + \frac{y}{\ln P_0}\right) + \mathcal{O}\left(\frac{H}{\ln T}\right), \\
 & x, y \in \left(0, \frac{\pi}{2}\right], \quad T \rightarrow \infty,
 \end{aligned}$$

where

$$(2.4) \quad \omega(t) = \frac{1}{\ln t} \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right) \right\},$$

and (see [1], (38);  $H \rightarrow U_1$ )

$$(2.5) \quad Q_1 = \sum_{T \leq t_\nu \leq T+U_1} 1 = \frac{1}{\pi} U_1 \ln P_0 + \mathcal{O}\left(\frac{U_1^2}{T}\right),$$

$$U_1 = \sqrt{T} \ln P_0, \quad P_0 = \sqrt{\frac{T}{2\pi}}.$$

Furthermore we have

$$(2.6) \quad m\{\mathring{G}_1(x)\}, \quad m\{\mathring{G}_2(y)\} < T^{1/3+\epsilon}.$$

2.2.

*Remark 2.* Every of the correlation integrals in (2.3) contains the product

$$(2.7) \quad Z[\varphi_1(t)]Z^2(t)$$

and, as usually, we have (see [6], (6.2); [7], (8.4))

$$(2.8) \quad t - \varphi_1(t) \sim (1-c) \frac{t}{\ln t} \sim (1-c)\pi(t), \quad t \rightarrow \infty,$$

i. e. we have big difference of arguments in (2.7), where  $c$  is the Euler constant and  $\pi(t)$  is the prime-counting function.

Next, in the case

$$t = \mathring{T}; \quad \varphi_1(\mathring{T}) = T \Rightarrow T \rightarrow \infty \Leftrightarrow \mathring{T} \rightarrow \infty$$

we obtain from (2.8)

$$\begin{aligned} \mathring{T} - T &\sim (1-c) \frac{\mathring{T}}{\ln \mathring{T}} \Rightarrow 1 - \frac{T}{\mathring{T}} \sim (1-c) \frac{1}{\ln \mathring{T}} \Rightarrow \\ &\Rightarrow \mathring{T} \sim T \Rightarrow \ln \mathring{T} \sim \ln T, \end{aligned}$$

i. e.

$$\mathring{T} - T \sim (1-c) \frac{T}{\ln T},$$

and (see (2.5))

$$\mathring{T} - (T + U_1) \sim (1-c) \frac{T}{\ln T} - U_1 \sim (1-c) \frac{T}{\ln T}.$$

Consequently we have

$$(2.9) \quad \rho\{\mathring{T}, \widehat{\mathring{T} + H}; [T, T + U_1]\} \sim (1-c)\pi(T);$$

$$[T, T + U_1] \prec \mathring{[T, \widehat{\mathring{T} + H}]},$$

where  $\rho$  stands for the distance of the corresponding segments. We may, of course, put

$$(2.10) \quad G_1(x) \cap [T, T + H] = \bar{G}_1(x) \rightarrow G_1(x), \dots$$

if necessary.

*Remark 3.* We have the following properties

(a)

$$\mathring{G}_1(x), \mathring{G}_2(y) \in [\mathring{T}, \widehat{\mathring{T} + H}] \rightarrow [T, T + U],$$

(comp. (1.3), (2.10)), where  $\mathring{G}_1(x), \mathring{G}_2(y)$  are disconnected sets,

(b) if

$$\mathcal{G} = \mathring{G}_1(x), \mathring{G}_2(y)$$

then extremely big distance occurs, namely (comp. (2.9), (2.10))

$$\rho\{\mathcal{G}; [T, T + U_1]\} \sim (1 - c)\pi(T), \quad T \rightarrow \infty,$$

(c) for the corresponding orders of  $Z$  (comp. (1.3))

$$1 + 2 \rightarrow 1 + 1$$

then the formulae (2.3) are *strongly heterogeneous* (comp. Remark 1).*Remark 4.* Moreover we explicitly notice the following:

- (a) the formulae (2.3) are not accessible by the current methods in the theory of the Riemann zeta-function,
- (b) small improvements of the exponents

$$\frac{1}{6}, \frac{1}{2}, \dots$$

are irrelevant for main direction of this paper (comp. our paper [2], Appendix A: On I.M. Vinogradov' scepticism on possibilities of the method of trigonometric sums).

### 3. BIG ASYMMETRY AND RELATED FORMULAE

3.1. Let (see [4], p. 29)

$$\begin{aligned}
 G_3(x) &= G_3(x; T, U_2) = \\
 &= \bigcup_{T \leq g_{2\nu} \leq T+U_2} \{t : g_{2\nu}(-x) < t < g_{2\nu}(x)\}, \quad 0 < x \leq \frac{\pi}{2}, \\
 (3.1) \quad G_4(y) &= G_4(y; T, U_2) = \\
 &= \bigcup_{T \leq g_{2\nu+1} \leq T+U_2} \{t : g_{2\nu+1}(-y) < t < g_{2\nu+1}(y)\}, \quad 0 < y \leq \frac{\pi}{2}, \\
 U_2 &= T^{5/12+2\epsilon},
 \end{aligned}$$

where the collection of sequences

$$\{g_\nu(\tau)\}, \quad \tau \in [-\pi, \pi], \quad \nu = 1, 2, \dots$$

is defined by the equation (see [4], (6))

$$\vartheta_1[g_\nu(\tau)] = \frac{\pi}{2}\nu + \frac{\tau}{2}; \quad g_\nu(0) = g_\nu,$$

where (comp. (1.2))

$$\vartheta(t) = \vartheta_1(t) + \mathcal{O}\left(\frac{1}{t}\right), \quad \vartheta_1(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8},$$

and (see [4], (13))

$$(3.2) \quad \begin{aligned} m\{G_3\} &= \frac{x}{\pi} U_2 + \mathcal{O}\left(\frac{x}{\ln T}\right), \\ m\{G_4\} &= \frac{y}{\pi} U_2 + \mathcal{O}\left(\frac{y}{\ln T}\right). \end{aligned}$$

Let

$$\varphi_1\{\mathring{G}_3(x)\} = G_3(x), \quad \varphi_1\{\mathring{G}_4(y)\} = G_4(y).$$

The following Theorem holds true.

**Theorem 2.**

$$(3.3) \quad \begin{aligned} &\int_{\mathring{G}_3(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt \sim \\ &\sim \frac{4}{\pi} U_2 \sin x + \int_{\mathring{G}_4(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt, \\ &x \in \left(0, \frac{\pi}{2}\right], \quad T \rightarrow \infty. \end{aligned}$$

*Remark 5.* By the asymptotic formula (3.3) is expressed the property of big asymmetry in the distribution of the values of  $Z$  on disconnected sets  $\mathring{G}_3(x), \mathring{G}_4(x)$ . Namely, the correlation integral (comp. (2.8)) on the disconnected set  $\mathring{G}_3(x)$  essentially exceeds of that on the set  $\mathring{G}_4(x)$ . For example

$$\int_{\mathring{G}_3(\pi/2)} - \int_{\mathring{G}_4(\pi/2)} \sim \frac{4}{\pi} U_2, \quad T \rightarrow \infty,$$

where

$$m\{\mathring{G}_3(\pi/2)\} + m\{\mathring{G}_4(\pi/2)\} = U_2$$

if (comp. (2.10))

$$G_3(x) \cap [T, T + U_2] \rightarrow G_3(x), \dots$$

3.2. Finally, the following Theorem holds true.

**Theorem 3.**

$$(3.4) \quad \begin{aligned} &\int_{\mathring{G}_3(x)} \omega(t) Z[\varphi_1(t)] Z^2(t) dt = \\ &= \frac{m\{G_1(x)\}}{2m\{G_3(x)\}} \int_{\mathring{G}_3(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt - \\ &- \frac{m\{G_1(x)\}}{2m\{G_4(x)\}} \int_{\mathring{G}_4(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt + \mathcal{O}(HT^{-\epsilon}), \\ &\int_{\mathring{G}_2(y)} \omega(t) Z[\varphi_1(t)] Z^2(t) dt = \\ &= \frac{m\{G_2(y)\}}{2m\{G_4(y)\}} \int_{\mathring{G}_4(y)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt - \\ &- \frac{m\{G_2(y)\}}{2m\{G_3(y)\}} \int_{\mathring{G}_3(y)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt + \mathcal{O}(HT^{-\epsilon}), \\ &x, y \in \left(0, \frac{\pi}{2}\right], \quad T \rightarrow \infty. \end{aligned}$$

*Remark 6.* For the disconnected sets

$$\dot{G}_1(x), \dot{G}_2(y), \dot{G}_3(x), \dot{G}_4(x), \dot{G}_3(y), \dot{G}_4(y); \dot{G}_1 \cap \dot{G}_2 = \emptyset, \dot{G}_3 \cap \dot{G}_4 = \emptyset$$

we have the following property: the correlation integrals of the order  $1 + 2$  on  $\dot{G}_1(x), \dot{G}_2(y)$  are expressed as the linear combinations of the correlation integrals of the order  $2 + 2$  on  $\dot{G}_3(x), \dot{G}_4(x); \dot{G}_3(y), \dot{G}_4(y)$  correspondingly.

#### 4. PROOF OF THEOREM 1

4.1. In the paper [1] (see (10)) we have proved the following autocorrelative formula

$$(4.1) \quad \begin{aligned} \sum_{T \leq t_\nu \leq T+U_1} Z(t_\nu) Z(t_\nu + \beta) &= \\ &= \frac{2 \sin(\beta \ln P_0)}{\pi \beta \ln P_0} U_1 \ln P_0 + \mathcal{O}(\sqrt{T} \ln^2 T), \end{aligned}$$

where

$$(4.2) \quad \beta = \mathcal{O}\left(\frac{1}{\ln T}\right), \quad U_1 = \sqrt{T} \ln P_0.$$

In the case

$$\beta \ln P_0 = x \Rightarrow \beta = \frac{x}{\ln P_0}, \quad x \in \left(0, \frac{\pi}{2}\right]$$

we obtain from (4.1), (4.2)

$$(4.3) \quad \begin{aligned} \sum_{T \leq t_\nu \leq T+U_1} Z(t_\nu) Z\left(t_\nu + \frac{x}{\ln P_0}\right) &= \\ &= \frac{2 \sin x}{\pi x} U_1 \ln^2 P_0 + \mathcal{O}(\sqrt{T} \ln^2 T). \end{aligned}$$

Consequently we have (see (2.5), (4.2))

$$(4.4) \quad \begin{aligned} \frac{1}{Q_1 \ln P_0} \sum_{T \leq t_\nu \leq T+U_1} Z(t_\nu) Z\left(t_\nu + \frac{x}{\ln P_0}\right) &= \\ &= 2 \frac{\sin x}{x} + \mathcal{O}\left(\frac{1}{\ln T}\right); \quad x \in \left(0, \frac{\pi}{2}\right] \Rightarrow \frac{\sin x}{x} \in \left[\frac{2}{\pi}, 1\right]. \end{aligned}$$

4.2. Next, in the paper [3], (5),(9) we have obtained the following mean-value formula on the disconnected sets  $G_1(x), G_2(y)$  (see (2.1))

$$(4.5) \quad \begin{aligned} \int_{G_1(x)} Z(t) dt &= \frac{2}{\pi} H \sin x + \mathcal{O}(x T^{1/6+\epsilon}), \\ \int_{G_2(y)} Z(t) dt &= -\frac{2}{\pi} H \sin y + \mathcal{O}(y T^{1/6+\epsilon}), \\ H &= T^{1/6+2\epsilon}; \quad T^{1/6} \psi^2 \ln^5 T \rightarrow T^{1/6+\epsilon}. \end{aligned}$$

Hence, from (4.4) by (2.3) we obtain

$$(4.6) \quad \begin{aligned} \frac{1}{m\{G_1(x)\}} \int_{G_1(x)} Z(t) dt &= 2 \frac{\sin x}{x} + \mathcal{O}(T^{-\epsilon}), \\ \frac{1}{m\{G_2(y)\}} \int_{G_2(y)} Z(t) dt &= -2 \frac{\sin y}{y} + \mathcal{O}(T^{-\epsilon}). \end{aligned}$$

4.3. In the paper [7], (9.2), (9.5) we have proved the following Lemma: if

$$\varphi_1\{\widehat{[T, T+U]}\} = [T, T+U],$$

then for every Lebesgue-integrable function

$$f(x), \quad x \in [T, T+U]$$

we have

$$(4.7) \quad \int_{\widehat{T}}^{\widehat{T+U}} f[\varphi_1(t)] \tilde{Z}^2(t) dt = \int_T^{T+U} f(x) dx,$$

$$T \geq T_0[\varphi_1], \quad U \in \left(0, \frac{T}{\ln T}\right],$$

where

$$(4.8) \quad \tilde{Z}^2(t) = \frac{Z^2(t)}{\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\} \ln t} = \omega(t) Z^2(t);$$

$$\omega(t) = \frac{1}{\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\} \ln t} = \frac{1}{\ln t} \left\{1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right\}.$$

Consequently, we have (see (4.7), (4.8))

$$(4.9) \quad \int_{\widehat{T}}^{\widehat{T+U}} \omega(t) f[\varphi_1(t)] Z^2(t) dt = \int_T^{T+U} f(x) dx,$$

$$U \in \left(0, \frac{T}{\ln T}\right].$$

4.4. Hence, by (4.6), (4.9) we obtain following formulae

$$(4.10) \quad \begin{aligned} & \frac{1}{m\{G_1(x)\}} \int_{\widehat{G_1(x)}} \omega(t) Z[\varphi_1(t)] Z^2(t) dt = \\ & = 2 \frac{\sin x}{x} + \mathcal{O}(T^{-\epsilon}), \\ & \frac{1}{m\{G_2(y)\}} \int_{\widehat{G_2(y)}} \omega(t) Z[\varphi_1(t)] Z^2(t) dt = \\ & = -2 \frac{\sin y}{y} + \mathcal{O}(T^{-\epsilon}). \end{aligned}$$

Finally, simple elimination of the values

$$2 \frac{\sin x}{x}, \quad 2 \frac{\sin y}{y}$$

from (4.4), (4.9) gives (2.3).

4.5. In the case

$$f(x) = 1$$

in (4.7) we obtain (see (2.1), (2.10), (4.8))

$$\begin{aligned} \int_{\mathring{G}_1(x)} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt &< \int_{\mathring{T}}^{\widehat{\mathring{T}+H}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \\ &\sim \ln T \int_T^{T+H} 1 \cdot dt, \end{aligned}$$

i. e.

$$(4.11) \quad \int_{\mathring{T}}^{\widehat{\mathring{T}+H}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt < AH \ln T, \quad T \rightarrow \infty.$$

Let

$$(4.12) \quad \widehat{\mathring{T}+H} - \mathring{T} \geq T^{1/3+\epsilon},$$

(comp. [6], (2.5); where  $\frac{1}{3}$  is the Balasubramanian exponent). Then

$$(4.13) \quad \int_{\mathring{T}}^{\widehat{\mathring{T}+H}} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim (\widehat{\mathring{T}+H} - \mathring{T}) \ln T > B(\widehat{\mathring{T}+H} - \mathring{T}) \ln T,$$

and (see (4.11), (4.13))

$$(4.14) \quad \widehat{\mathring{T}+H} - \mathring{T} < CH = CT^{1/6+\epsilon}.$$

Now, (4.14) contradicts (4.12). Consequently we have that

$$m\{\mathring{G}_1(x)\} < \widehat{\mathring{T}+H} - \mathring{T} < T^{1/3+\epsilon}$$

and we obtain the second inequality in (2.6) by the similar way.

## 5. PROOFS OF THEOREM 2 AND THEOREM 3

5.1. In the paper [4], (14), (15) we have proved the following mean-value formulae

$$\begin{aligned} (5.1) \quad \int_{G_3(x)} Z^2(t) dt &= \\ &= \frac{x}{\pi} U_2 \ln \frac{T}{2\pi} + \frac{2}{\pi} (cx + \sin x) U_2 + \mathcal{O}(T^{5/12} \ln^2 T), \\ \int_{G_4(y)} Z^2(t) dt &= \\ &= \frac{y}{\pi} U_2 \ln \frac{T}{2\pi} + \frac{2}{\pi} (cy - \sin y) U_2 + \mathcal{O}(T^{5/12} \ln^2 T) \end{aligned}$$



on the disconnected sets  $G_3(x), G_4(y)$ , (comp. (3.1)). From (5.1) we have in the case  $x = y$

$$\begin{aligned} \int_{G_3(x)} Z^2(t) dt - \int_{G_4(x)} Z^2(t) dt &= \\ &= \frac{4}{\pi} U_2 \sin x + \mathcal{O}(xT^{5/12} \ln^2 T). \end{aligned}$$

Consequently, we obtain from (5.1) by (4.7) – (4.9) with  $f(x) = Z^2(x)$  the formula (3.3).

5.2. Next, from the formula (see [4], (16))

$$\begin{aligned} \frac{1}{m\{G_3(x)\}} \int_{G_3(x)} Z^2(t) dt - \frac{1}{m\{G_4(x)\}} \int_{G_4(x)} Z^2(t) dt &\sim \\ &\sim 4 \frac{\sin x}{x}, \quad T \rightarrow \infty \end{aligned}$$

we obtain by (4.7) – (4.9) ( $f(x) = Z^2(x)$ ) the following formula

$$\begin{aligned} (5.2) \quad &\frac{1}{m\{G_3(x)\}} \int_{\hat{G}_3(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt - \\ &- \frac{1}{m\{G_4(x)\}} \int_{\hat{G}_4(x)} \omega(t) Z^2[\varphi_1(t)] Z^2(t) dt \sim 4 \frac{\sin x}{x}, \quad T \rightarrow \infty \end{aligned}$$

and, by the similar way, we obtain the formula for  $y \in (0, \pi/2]$ . Hence, the elimination of

$$2 \frac{\sin x}{x}, \quad 2 \frac{\sin y}{y}$$

from (4.10), (5.2) implies the formula (3.4).

I would like to thank Michal Demetrian for his help with electronic version of this paper.

#### REFERENCES

- [1] J. Moser, ‘On autocorrelative sum in the theory of the Riemann zeta-function’, Acta Math. Univ. Comen., 37 (1980), 121-134, (in Russian).
- [2] J. Moser, ‘On the roots of the equation  $Z'(t) = 0$ ’, Acta Arith. 40 (1981), 97-107, (in Russian), arXiv: 1303.0967.
- [3] J. Moser, ‘New consequences of the Riemann-Siegel formula’, Acta Arith., 42 (1982), 1-10, (in Russian), arXiv: 1312.4767.
- [4] J. Moser, ‘New mean-value theorems for the function  $|\zeta(\frac{1}{2} + it)|^2$ ’, Acta Math. Univ. Comen., 46-47 (1985), 21-40.
- [5] J. Moser, ‘On the order of Titchmarsh sum in the theory of the Riemann’s zeta-function and on the biquadratic effect in the information theory’, Czechoslovak Math. J., 41 (116), (1991), 663-684, (in Russian), arXiv: 1112.9548.
- [6] J. Moser, ‘Jacob’s ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral’, Math. Notes 88, (2010) 414-422, arXiv: 0901.3937.
- [7] J. Moser, ‘Jacob’s ladders, the structure of the Hardy-Littlewood integral and some new class of nonlinear integral equations’, Proc. Stek. Inst. 276, (2011), 208-221, arXiv: 1103.0359.
- [8] E.C. Titchmarsh, ‘On van der Corput’s method and the zeta-function of Riemann (IV)’, Quart. J. Math., (Oxford), 5, (1934), 98-105.
- [9] E.C. Titchmarsh, ‘The theory of the Riemann zeta-function’ Clarendon Press, Oxford, 1951.

DEPARTMENT OF MATHEMATICAL ANALYSIS AND NUMERICAL MATHEMATICS, COMENIUS UNIVERSITY, MLYNSKA DOLINA M105, 842 48 BRATISLAVA, SLOVAKIA

E-mail address: jan.moser@fmph.uniba.sk